Problem Set — Week 3

Introduction to Quantum Computing

Notes:

1. All of these problems can be solved via chatGPT, but I strongly encourage you to try solve them without using chatGPT!
2. If there is anything you don't understand, email a tutor to make an appointment, or bring your question to the following Q and A session:
   (a) Thursday, 2pm to 5:30pm, with Ryan.
3. If you really understand how to do all the problems in the problem sets, the final exam will be very easy for you.
4. This problem set will not be graded, but it is compulsory to submit solutions – i.e. in order to pass the course, you need to submit solutions to every problem set.
5. The deadline for submitting these solution is **10pm on Thursday the 23rd of October**. to submit the solutions, you should respond to the email that you receive from assignments@aims.ac.za before the deadline.

Important states and operators:

Description	Object	Dirac form	Matrix / vector
Computational basis	$\vert 0\rangle$	$\vert 0\rangle$	$\begin{bmatrix}1\\0\end{bmatrix}$
	$\vert 1\rangle$	$\vert 1\rangle$	$\begin{bmatrix}0\\1\end{bmatrix}$
X-basis states	$\vert +\rangle$	$\frac{1}{\sqrt{2}}(\vert 0\rangle + \vert 1\rangle)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$
	$\vert -\rangle$	$\frac{1}{\sqrt{2}}(\vert 0\rangle - \vert 1\rangle)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$
Y-basis states	$\vert +i\rangle$	$\frac{1}{\sqrt{2}}(\vert 0\rangle + i\,\vert 1\rangle)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$
	$\vert -i\rangle$	$\frac{1}{\sqrt{2}}(\vert 0\rangle - i\,\vert 1\rangle)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$
Identity	I	$\vert 0\rangle\langle 0\vert + \vert 1\rangle\langle 1\vert$	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$
Pauli X	X	$\vert 0\rangle\langle 1\vert + \vert 1\rangle\langle 0\vert$	$\begin{bmatrix}0&1\\1&0\end{bmatrix}$
Pauli Y	Y	$-i\,\vert 0\rangle\langle 1\vert + i\,\vert 1\rangle\langle 0\vert$	$\begin{bmatrix}0&-i\\i&0\end{bmatrix}$
Pauli Z	Z	$\vert 0\rangle\langle 0\vert - \vert 1\rangle\langle 1\vert$	$\begin{bmatrix}1&0\\0&-1\end{bmatrix}$
Hadamard gate	H	$\frac{1}{\sqrt{2}}(\vert 0\rangle\langle 0\vert + \vert 0\rangle\langle 1\vert + \vert 1\rangle\langle 0\vert - \vert 1\rangle\langle 1\vert)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}$

(For those submitting their homework using LaTeX, we suggest using the package `braket` *for the Dirac notation.)*

Question 1 (Density Matrices).

1. Which of the following are valid density matrices. For the ones that are not valid, why not?

$$\rho_0 = \begin{pmatrix} \frac{1}{4} & \frac{i}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix}, \qquad \rho_1 = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{2} & \frac{3}{4} \end{pmatrix}, \qquad \rho_2 = \begin{pmatrix} 1 & +i \\ -i & 0 \end{pmatrix}, \qquad \rho_4 = \begin{pmatrix} \frac{1}{8} & 0 \\ 0 & \frac{7}{8} \end{pmatrix} \tag{1}$$

2. Prove that for all square matrices A and B of the same dimension we have $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

3. Let ρ be a density operator. Prove that $\mathrm{Tr}(\rho^2) \leq 1$ with equality if and only if ρ is a pure state.

4. For each of the four Bell states, find the reduced density operator for each qubit.

5. Consider a single qubit system in the state

$$\rho = \begin{pmatrix} \frac{1}{2}, \frac{1}{2} \\ \frac{1}{2}, \frac{1}{2} \end{pmatrix}.$$

   What is the final state after the transformation via $U = X$?

6. Suppose a composite of systems A and B is in the state $|a\rangle|b\rangle$ where $|a\rangle$ is a pure state of system A and $|b\rangle$ is a pure state of system B. Show that the reduced density operator of system A alone is a pure state.

Question 2 (Matrix exponential).

1. Given two matrices A and B, prove that $[A, B] = 0$ implies $e^{A+B} = e^A e^B$.

2. Give an example of two 2×2 matrices C and D for which $e^{C+D} \neq e^C e^D$.

Question 3 (Pauli Matrices).

1. Calculate σ_i^2 for all $i \in \{1, 2, 3\}$ (Remember, $\sigma_1 = X$, $\sigma_2 = Y$, $\sigma_3 = Z$)

2. Calculate $[\sigma_i, \sigma_j]$ for all $(i, j) \in \{1, 2, 3\} \times \{1, 2, 3\}$.

3. Let ϵ_{ijk} be the Levi-Civita symbol. By using this symbol write down a simple expression for $[\sigma_i, \sigma_j]$ which holds for all $(i, j) \in \{1, 2, 3\} \times \{1, 2, 3\}$.

Question 4 (Hamiltonian time evolution).

1. Consider a quantum system described by a time-independent Hamiltonian H. Let the state of the system at time t be denoted by $|\psi(t)\rangle$. Let $0 \leq t_1 \leq t_2$. Write down the unitary operator $U(t_1, t_2)$ which maps $|\psi(t_1)\rangle$ to $|\psi(t_2)\rangle$.

2. Prove that the operator you wrote down in 2.1 is unitary.

Question 5 (Observables).

1. Let M be an observable, and let $M|\psi\rangle = m|\psi\rangle$. What is the average value observed when measuring M on state $|\psi\rangle$. What is the standard deviation?

2. Suppose we have a qubit in the state $|0\rangle$ and we measure the observable X. What is the average value of the measurement outcomes? What is the standard deviation?

3. Given a unit vector $\vec{v}$, prove that $\vec{v} \cdot \vec{\sigma}$ is a valid observable.

4. Calculate the probability of obtaining the result $+1$ for a measurement of $\vec{v} \cdot \vec{\sigma}$ given that the state prior to the measurement is $|0\rangle$. What is the state after the measurement if the result $+1$ was obtained?

5. Calculate the average value of the observable $X_1 Z_2$ when measuring the state $1/\sqrt{2}(|00\rangle + |11\rangle)$

Questions 6 and 7 (Deutsch And Grover Algorithm).

See the attached Jupyter notebooks!