Problem Set — Week 2

Introduction to Quantum Computing

Notes:

1. All of these problems can be solved via chatGPT, but I strongly encourage you to try solve them without using chatGPT!
2. If there is anything you don't understand, email a tutor to make an appointment, or bring your question to one of the following two question and answer sessions:
   (a) Thursday, 3pm to 5pm, with Ryan.
   (b) Friday, 4pm to 6pm, with Florian.
3. If you really understand how to do all the problems in the problem sets, the final exam will be very easy for you.
4. This problem set will not be graded, but it is compulsory to submit solutions – i.e. in order to pass the course, you need to submit solutions to every problem set.
5. The deadline for submitting these solution is **10pm on Saturday the 18th of October**. to submit the solutions, you should respond to the email that you receive from assignments@aims.ac.za before the deadline.

Important states and operators:

Description	Object	Dirac form	Matrix / vector
Computational basis	$\|0\rangle$	$\|0\rangle$	$\begin{bmatrix}1\\0\end{bmatrix}$
	$\|1\rangle$	$\|1\rangle$	$\begin{bmatrix}0\\1\end{bmatrix}$
X-basis states	$\|+\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$
	$\|-\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$
Y-basis states	$\|+i\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle + i\,\|1\rangle)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$
	$\|-i\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle - i\,\|1\rangle)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$
Identity	I	$\|0\rangle\langle 0\| + \|1\rangle\langle 1\|$	$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$
Pauli X	X	$\|0\rangle\langle 1\| + \|1\rangle\langle 0\|$	$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$
Pauli Y	Y	$-i\,\|0\rangle\langle 1\| + i\,\|1\rangle\langle 0\|$	$\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$
Pauli Z	Z	$\|0\rangle\langle 0\| - \|1\rangle\langle 1\|$	$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$
Hadamard gate	H	$\frac{1}{\sqrt{2}}(\|0\rangle\langle 0\| + \|0\rangle\langle 1\| + \|1\rangle\langle 0\| - \|1\rangle\langle 1\|)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$

(For those submitting their homework using LaTeX, we suggest using the package `braket` *for the Dirac notation.)*

Question 1 (Tensor products and the conjugate transpose).

1. Let
$$A = \begin{pmatrix} 1 & i \\ i & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} i & 1 \\ 0 & 1-i \end{pmatrix}.$$
Compute the tensor product $A \otimes B$ explicitly as a 4×4 matrix. Then compute $(A \otimes B)^\dagger$. Verify by direct computation that
$$(A \otimes B)^\dagger \;=\; A^\dagger \otimes B^\dagger.$$

2. Prove the following identity for matrices $A, C \in \mathbb{C}^{n\times n}$ and $B, D \in \mathbb{C}^{m\times m}$,
$$(A \otimes B)(C \otimes D) \;=\; (AC) \otimes (BD).$$

3. Let $|\psi\rangle = \frac{1}{\sqrt{3}}\big(|00\rangle + i\,|01\rangle + |10\rangle\big)$.
   (a) Write $|\psi\rangle$ as a column vector in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.
   (b) Compute the conjugate transpose $|\psi\rangle^\dagger = \langle\psi|$.
   (c) Form the operator $\rho = |\psi\rangle\langle\psi|$. Show that ρ is Hermitian and satisfies $\rho^2 = \rho$.

4. Let
$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$
   (a) Show that U and V are unitary by verifying $U^\dagger U = V^\dagger V = \mathbb{1}$.
   (b) Compute $U \otimes V$ and verify that it is also unitary by checking $(U \otimes V)^\dagger (U \otimes V) = \mathbb{1}$.

5. Let $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Compute the 4×4 matrices for $X \otimes \mathbb{1}$ and $\mathbb{1} \otimes Z$ and verify via explicit calculation that
$$(X \otimes \mathbb{1})(\mathbb{1} \otimes Z) = X \otimes Z.$$

Question 2 (Projective measurements).

Here is a reminder of some terminology and definitions:

1. A *(orthogonal) projector* on a finite-dimensional Hilbert space $\mathcal{H}$ is a linear operator P satisfying $P^\dagger = P$ and $P^2 = P$.

2. A *projective measurement* (PVM) is a finite collection $\{P_j\}_j$ of projectors on $\mathcal{H}$ such that $P_j P_k = \delta_{jk} P_j$ and $\sum_j P_j = \mathbb{1}$.

3. For a pure state $|\psi\rangle$, the outcome probabilities when measuring $|\psi\rangle$ with PVM $\{P_j\}_j$ are $p_j = \langle\psi|P_j|\psi\rangle$ and the post-measurement (normalized) state upon outcome j is $|\psi_j\rangle = \frac{P_j|\psi\rangle}{\sqrt{p_j}}$.

With this in mind, answer the following questions:

1. *Basic checks.*

   (a) Decide whether each of the following matrices is a projector. If it is, find its range (the subspace onto which it projects).

   $$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad P_3 = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

   (b) Show that $\{P_2,\ \mathbb{1} - P_2\}$ is a valid projective measurement on a single qubit. What are the corresponding measurement subspaces?

   (c) Let $|v_\theta\rangle = \cos\theta\ |0\rangle + \sin\theta\ |1\rangle$ and $|v_\theta^\perp\rangle = -\sin\theta\ |0\rangle + \cos\theta\ |1\rangle$. Define $P_\theta = |v_\theta\rangle\langle v_\theta|$ and $P_\theta^\perp = |v_\theta^\perp\rangle\langle v_\theta^\perp|$. Show that $\{P_\theta, P_\theta^\perp\}$ is a projective measurement for all θ.

2. *Calculations with probabilities and post-measurement states.*

   (a) Let $P_+ = |+\rangle\langle+|$ and $P_- = |-\rangle\langle-|$ with $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. For

   $$|\psi\rangle = \frac{\sqrt{3}}{2}\,|0\rangle + \frac{1}{2}\,e^{i\pi/3}\,|1\rangle\,,$$

   compute $p_\pm = \langle\psi|P_\pm|\psi\rangle$ and the post-measurement states $|\psi_\pm\rangle = \frac{P_\pm|\psi\rangle}{\sqrt{p_\pm}}$.

   (b) Consider the four-outcome PVM

   $$\mathcal{M}_{XZ} = \big\{|a\rangle\langle a| \otimes |b\rangle\langle b| \text{ with } a \in \{+,-\} \text{ and } b \in \{0,1\}\big\}.$$

   For the state

   $$|\phi\rangle = \frac{1}{\sqrt{6}}\big(2\,|00\rangle + |01\rangle + |10\rangle\big),$$

   compute the four outcome probabilities and the corresponding normalized post-measurement states.

   (c) Let $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and measure *only the first qubit* in the X-basis with projectors

   $$\Pi_+ = |+\rangle\langle+| \otimes \mathbb{1}, \qquad \Pi_- = |-\rangle\langle-| \otimes \mathbb{1}.$$

   Compute $p_\pm = \langle\Phi^+|\Pi_\pm|\Phi^+\rangle$ and the (normalized) post-measurement states $(\Pi_\pm\,|\Phi^+\rangle)/\sqrt{p_\pm}$ for the *two-qubit system*.

Question 3 (Hermitian operators define projective measurements).

Consider the following three Hermitian matrices.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad C = \begin{pmatrix} 3 & 1 & i \\ 1 & 3 & i \\ -i & -i & 5 \end{pmatrix}.$$

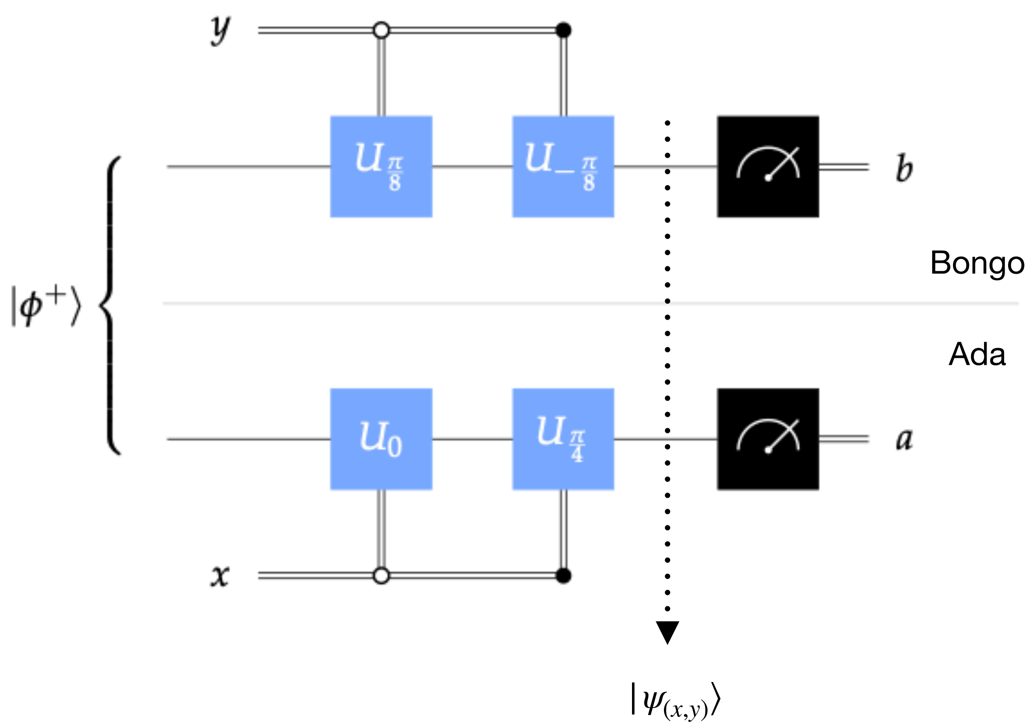
For each matrix $M \in \{A, B, C\}$, do the following.

1. Compute all eigenvalues and a corresponding orthonormal set of eigenvectors (also called eigenstates).

2. For each eigenvalue λ, define the projector P_λ onto its (one-dimensional) eigenspace, i.e. $P_\lambda = |v_\lambda\rangle\langle v_\lambda|$ for a normalized eigenvector $|v_\lambda\rangle$.
3. Verify (by direct calculation) that each P_λ satisfies the projector properties.
4. Pick two different states $|\psi\rangle$, decompose each in the eigenbasis of M, and compute $P_\lambda |\psi\rangle$ to see which components remain.
5. Show that $\{P_\lambda\}$ forms a valid projective measurement by checking $\sum_\lambda P_\lambda = \mathbb{1}$ and $P_\lambda P_\mu = \delta_{\lambda\mu} P_\lambda$. Conclude that $M = \sum_\lambda \lambda P_\lambda$ is the spectral decomposition (i.e. measuring M corresponds to this PVM).

Question 4 (A winning quantum strategy for the CHSH game.).

In this exercise we are going to complete the proof that the quantum strategy for the CHSH game (that we discussed in class on Monday), wins with probability approximately 85%. To this end, recall the following circuit diagram, which describes the strategy for Bongo and Ada:



Additionally, remember that Bongo and Ada win when $a \oplus b = x \wedge y$.

Finally, recall the following definitions:

$$|\psi_\theta\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$$
$$U_\theta = |0\rangle\langle\psi_\theta| + |1\rangle\langle\psi_{\theta+\pi/2}|$$

1. *Warm up:*

   (a) Prove that $U_\theta U_\theta^\dagger = \mathbb{1}$

   (b) Prove that

$$(\langle\psi_\alpha|\langle\psi_\beta|)\,|\phi^+\rangle = \frac{\cos(\alpha-\beta)}{\sqrt{2}}. \tag{1}$$

2. *Winning probabilities.* For each possible question $(x, y) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$:

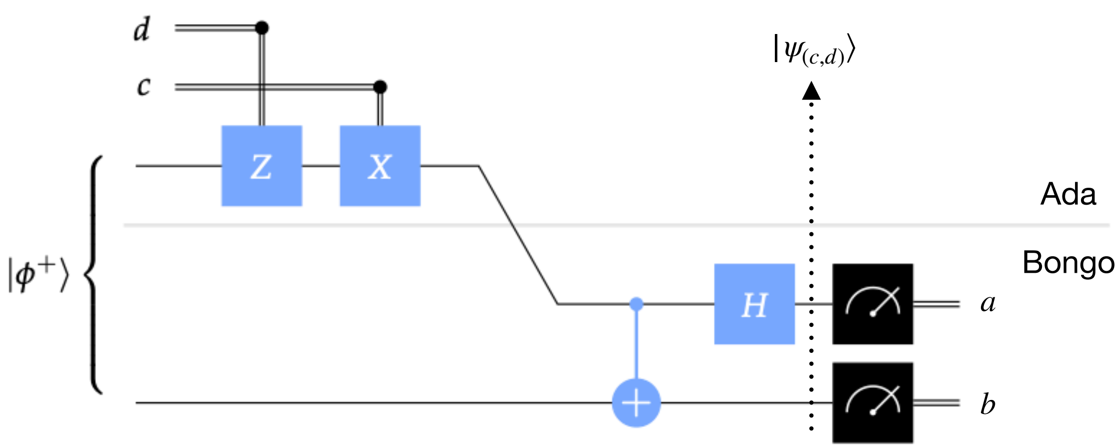
   (a) Calculate the state $|\psi_{(x,y)}\rangle$ – *Hint: Use Eq.* (1)*!*

   (b) For each possible set of answers $(a, b) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ calculate the probability of obtaining the outcome (a, b) when measuring the state $|\psi_{(x,y)}\rangle$ with a standard basis measurement.

   (c) Calculate the probability of Bongo and Ada winning the game for this specific question.

Question 5 (Superdense Coding).

Once again, lets think about Ada in her lab in Abuja, and Bongo in his lab in Nairobi. This time around, Ada wants to send two *classical* bits to Bongo, but she has run out of data for sending emails, and can only send *one qubit* to Bongo. This seems impossible. Luckily, her and Bongo have already bought an e-bit from the entanglement factory, and she has an idea! In particular, she proposes for her and Bongo to execute the following quantum circuit:



1. Describe in words the protocol depicted in the circuit.

2. For all possible bit combinations $(c, d) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$:

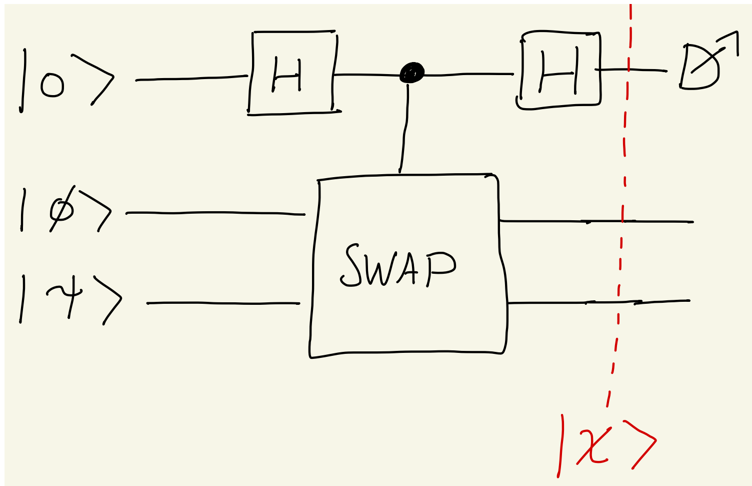
   (a) Calculate the state $|\psi_{(c,d)}\rangle$.

   (b) For all possible measurement outcomes $(a, b) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ calculate the probability of obtaining (a, b)

3. Does this circuit indeed do what Ada wanted?

Question 7 (SWAP test 2.0).

Consider the following quantum circuit:



1. Describe in words what the controlled-SWAP gate in the middle of the circuit does.
2. Calculate the state $|\chi\rangle$ at the end of the circuit, before measurement.
3. Write down the probability of obtaining "0" when measuring the top qubit with a standard basis measurement.
4. What can we use this circuit for?

Question 8 (Asymptotic complexity of integer addition).

In class I promised you that we would explore the family of classical circuits – i.e. the algorithm – for the problem of *integer addition* - so, here it is! To answer this question, I strongly suggest you read Lesson 6 of the textbook carefully!

1. Write down a classical circuit – using only FANOUT, NOT, AND and OR gates – for adding together two two-bit integers. How many elementary gates are in this circuit?
2. Write down a classical circuit for adding together two three-bit integers. To make this easier to draw, here you can use some "sub-routine" gates. How many elementary gates are in this circuit?
3. Write down the function $f(n)$ which describes the number of gates in the circuit for adding together two n-bit integers.
4. Write down a very simple function $g(n)$ for which $f(n) = O(g(n))$.