Problem Set — Week 1: Fundamentals of Quantum Computing

Introduction to Quantum Computing

Notes:

1. All of these problems can be solved via chatGPT, but I strongly encourage you to try solve them without using chatGPT!
2. If there is anything you don't understand, email a tutor to make an appointment, or bring your question to the question and answer session on Thursday 3pm to 5pm.
3. If you really understand how to do all the problems in the problem sets, the final exam will be very easy for you.
4. This problem set will not be graded, but it is compulsory to submit solutions – i.e. in order to pass the course, you need to submit solutions to every problem set.
5. The deadline for submitting these solution is **5pm on Saturday the 11th of October**. to submit the solutions, you should respond to the email that you receive from assignments@aims.ac.za before the deadline.

Important states and operators:

Description	Object	Dirac form	Matrix / vector
Computational basis	$\|0\rangle$	$\|0\rangle$	$\begin{bmatrix}1\\0\end{bmatrix}$
	$\|1\rangle$	$\|1\rangle$	$\begin{bmatrix}0\\1\end{bmatrix}$
X-basis states	$\|+\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$
	$\|-\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$
Y-basis states	$\|+i\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle + i\,\|1\rangle)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$
	$\|-i\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle - i\,\|1\rangle)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$
Identity	I	$\|0\rangle\langle 0\| + \|1\rangle\langle 1\|$	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$
Pauli X	X	$\|0\rangle\langle 1\| + \|1\rangle\langle 0\|$	$\begin{bmatrix}0&1\\1&0\end{bmatrix}$
Pauli Y	Y	$-i\,\|0\rangle\langle 1\| + i\,\|1\rangle\langle 0\|$	$\begin{bmatrix}0&-i\\i&0\end{bmatrix}$
Pauli Z	Z	$\|0\rangle\langle 0\| - \|1\rangle\langle 1\|$	$\begin{bmatrix}1&0\\0&-1\end{bmatrix}$
Hadamard gate	H	$\frac{1}{\sqrt{2}}(\|0\rangle\langle 0\| + \|0\rangle\langle 1\| + \|1\rangle\langle 0\| - \|1\rangle\langle 1\|)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}$

(For those submitting their homework using LaTeX, we suggest using the package `braket` *for the Dirac notation.)*

Question 1 (Two-level system, $\Sigma = \{0, 1\}$).

1. Write each of the following column vectors in Dirac notation:
$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}, \frac{1}{\sqrt{10}}\begin{bmatrix}3\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}i\\1\end{bmatrix}.$

2. Write each of the following Dirac kets as column vectors (in the computational basis):
$|0\rangle\,,\ |+\rangle\,,\ |-i\rangle\,,\ |\theta\rangle = \frac{1}{2}\big(\sqrt{3}\,|0\rangle + |1\rangle\big)\,,\ |\eta\rangle = \frac{1}{\sqrt{5}}(2\,|0\rangle - i\,|1\rangle)\,,\ |\xi\rangle = \frac{1}{\sqrt{10}}(|-\rangle + 3\,|+\rangle)$

3. Compute the following inner products either using column vectors or Dirac notation:
$\langle -|+\rangle\,,\ \langle +|+i\rangle\,,\ \langle +i|+\rangle\,,\ \langle -i|0\rangle\,,\ \langle +|\theta\rangle\,,\ \langle \xi|\eta\rangle.$

4. Compute the matrix form for the following outer products:
$|0\rangle\langle 0|\,,\ |+\rangle\langle +|\,,\ |1\rangle\langle +|\,,\ |\eta\rangle\langle\eta|\,,\ |\xi\rangle\langle\theta|.$

5. Let $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ with $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$. Compute $X\,|\psi\rangle\,,\ Y\,|\psi\rangle\,,\ Z\,|\psi\rangle\,,\ H\,|\psi\rangle$, and give the results both in Dirac notation and as column vectors.

6. Using the results from the previous question, say what the probability is of obtaining the state $|0\rangle$ when measuring $X\,|\psi\rangle\,,\ Y\,|\psi\rangle\,,\ Z\,|\psi\rangle\,,\ H\,|\psi\rangle$.

7. Given any two valid qubit states $|\psi\rangle$ and $|\phi\rangle$, what is the relation between $\langle\psi|\phi\rangle$ and $\langle\phi|\psi\rangle$?

8. Rewrite the following matrices using kets and bras in the $\{|0\rangle, |1\rangle\}$ basis (i.e. express each as a linear combination of $|j\rangle\langle k|$ for $j, k \in \{0, 1\}$.):
$X,\ Y,\ Z,\ H$

9. Prove that the matrices Y, Z and H are unitary.

10. Prove that both $|+\rangle$ and $|-\rangle$ are eigenstates of X.

11. For each of the following, say whether it is a valid quantum state, classical state, or neither:
$|\psi\rangle = \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle,\ |\psi\rangle = \frac{1}{3}|0\rangle - \frac{2i}{3}|1\rangle, |\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}i}{\sqrt{3}}|1\rangle, |\psi\rangle = \sqrt{2}|0\rangle - |+\rangle$

Question 2 (Five-level system, $\Sigma = \{0, 1, 2, 3, 4\}$).
Let $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle, |4\rangle\}$ be the computational basis of the five-level system.

1. Write each of the following column vectors in Dirac notation:
$$\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\\0\end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}0\\e^{i\pi/4}\\e^{-i\pi/4}\\0\\0\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}2\\-1\\0\\i\\0\end{bmatrix}.$$

2. Write each of the following Dirac-Kets as column vectors (in the computational basis):
$|0\rangle\,,\ |4\rangle\,,\ |\nu\rangle = \frac{1}{\sqrt{5}}\sum_{k=0}^{4}|k\rangle\,,\ |\omega\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |3\rangle),\ |\mu\rangle = \frac{1}{\sqrt{5}}\sum_{k=0}^{4} \mathrm{e}^{2\pi i\frac{k}{5}}\,|k\rangle.$

3. Compute the following inner products either using column vectors or Dirac notation:
$\langle 3|4\rangle\,,\ \langle 2|\nu\rangle\,,\ \langle\nu|\omega\rangle\,,\ \langle\nu|\mu\rangle.$

4. Compute the matrix form for the following outer products:
$|1\rangle\langle 3|\,,\ |0\rangle\langle\omega|\,,\ |\mu\rangle\langle\mu|.$

Question 3 (Tensor products). Assume we have several independent two-level systems, each with configuration set $\Sigma = \{0, 1\}$. Use lexicographic order for the computational basis.

1. Compute the column vector for each of the following states: $|0\rangle \otimes |+\rangle$, $|+i\rangle \otimes |1\rangle$, $|+\rangle \otimes |+\rangle \otimes |+i\rangle$.
2. Let $|\chi\rangle = \frac{1}{2}|01\rangle - \frac{\sqrt{3}}{2}|10\rangle$. Write $|\chi\rangle$ as column vector and compute the matrix form for the outer product $|\chi\rangle\langle\chi|$.
3. Show that $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is not a product state by attempting to factor it as $|a\rangle \otimes |b\rangle$.
4. Compute the 8×8 matrix form of $X \otimes H \otimes Z$.

Question 4 (Applying tensor-product operators).

1. Compute $X|0\rangle$, $X|1\rangle$, $Z|0\rangle$, $Z|1\rangle$ and give the results both in Dirac notation and as column vectors.
2. Compute the matrix form of $X \otimes Z$ and apply it to the column vectors corresponding to the four computational-basis states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$. Redo the same calculation using only Dirac notation.
3. Let $|\phi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$ with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$. Compute $(X \otimes Z)|\phi\rangle$ in terms of $\alpha, \beta, \gamma, \delta$, both using Dirac notation and column vectors.
4. How does the action X and Z on single qubits connect to the action of $(X \otimes Z)$ on the two-qubit system?

Question 5 (Circuits and gadgets).

1. Describe the action of the following two-qubit gates:
$$\mathsf{SWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathsf{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathsf{CZ} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$
Explain how these differ from tensor-product (separable) single-system gates.
2. Starting from $|00\rangle$, design a circuit that prepares $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Specify the gate sequence.
3. Construct SWAP using a sequence of CNOT and single-qubit gates, and show equivalence explicitly by calculating the matrix form of your circuit.
4. Explain the SWAP-test circuit and how it estimates the overlap $|\langle\psi|\phi\rangle|^2$ of two states $|\psi\rangle$ and $|\phi\rangle$. Indicate the role of measurement statistics and the ancilla qubit.

Question 6 (Qiskit exploration). Later in this course, we might (hopefully!) be able to run quantum circuits on real quantum computers provided by IBM. As preparation, go through all the jupyter notebooks in the Basics of quantum information section of the Fundamentals of Quantum Algorithms course by IBM and make sure to understand everything :) This should also help you understand the material better.